Algorithmic Game Theory Due: April 17, 2018

Homework 1

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1 Auctions

Problem 1: Consider a second price auction with n bidders and suppose a subset S of the bidders decide to collude, meaning that they submit false bids in a coordinated way to maximize the sum of their payoffs. Establish necessary and sufficient conditions on the set S (in terms of the private valuations of the bidders) such that the bidders of S can increase their collective payoff via non-truthful bidding.

Problem 2: Consider a combinatorial auction with a set M of m goods and n bidders. Assume that the valuation function of every bidder $v_i(\cdot)$ is normalized, monotone, and *subadditive* (i.e., for every disjoint sets T_1, T_2, $v_i(T_1) + v_i(T_2) \geq v_i(T_1 \cup T_2)$).

For now, ignore payments and truthfulness, rather consider only poly-time social welfare maximization. Given M and $v_1, \ldots, v_n$, call the setting *lopsided* if there is an optimal allocation of goods in which at least half of the total SW of the allocation is due to players that were allocated a bundle with at least $\sqrt{m}$ goods. (i.e., if $\sum_{i \in A} v_i(T_i^*) \geq \frac{1}{2} \sum_{i=1}^{n} v_i(T_i^*)$, where T^* is the optimal allocation and A is the subset of bidders i with $|T_i^*| \geq \sqrt{m}$.)

1. Show that in a lopsided problem, there is an allocation that gives all the goods to a single player and achieves an $\Omega(1/\sqrt{m})$ fraction of the maximum-possible SW.
2. Show that in a problem that is not lopsided, there is an allocation that gives at most one good to each player and achieves an $\Omega(1/\sqrt{m})$ fraction of the maximum-possible SW. [hint: use subadditivity.]
3. Give a poly-time $O(\sqrt{m})$-approximate algorithm for subadditive valuations. [hint: make use of a graph matching algorithm].
4. Give a poly-time $O(\sqrt{m})$-approximate, truthful combinatorial auction for subadditive valuations.

2 Sponsored Search

Problem 3: Consider the sponsored search setting: there are k slots for advertisements on a search results page, and the bidders are advertisers that want to use these slots for advertisements. The slots are not identical - slots that appear higher on the search results page are usually more valuable than slots that appear lower on the page. The difference between the slots is measured by a click through rate (CTR) value: for each slot j its CTR is α_j, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_k$. α_j represent the probability that the user will click on slot j. Advertiser i has a private valuation v_i for each click on her advertisement. The value that advertiser i gains from slot j is $v_i \alpha_j$. An auction is done to determine how the slots will be assigned.

In the current problem, we consider an extension of the sponsored search setting. Each bidder i now has a publicly known *quality* β_i (in addition to a private valuation v_i per click). As usual, each slot j has a click-through-rate (CTR) α_j, and $\alpha_1 \geq \alpha_2 \ldots \geq \alpha_k$. We assume that if bidder i is placed in slot j, its probability of a click is $\beta_i \alpha_j$ – thus, bidder i derives value $v_i \beta_i \alpha_j$ from this outcome.

Describe the surplus-maximizing allocation rule in this generalized sponsored search setting. Argue that this rule is monotone. Give an explicit formula for the per-click payment of each bidder that extends this allocation rule to a DSIC mechanism.

3 VCG

Problem 4: Consider a combinatorial auction in which a bidder can submit multiple bids under different names, unbeknownst to the mechanism. The allocation and payment of a bidder is the union and sum of the allocations and payments, respectively assigned to all of its pseudonyms. Prove or disprove:

1. a bidder in a combinatorial auction can earn higher utility from the VCG mechanism by submitting multiple bids than by bidding truthfully.
2. a bidder in a single item auction can earn higher utility from the second price auction mechanism by submitting multiple bids than by bidding truthfully.

4 Online bipartite matching

Problem 5: Let $G = (L, R, E)$ be a bipartite graph where $|R| = |L| = n$. Consider the online bipartite matching problem as was defined in the lecture. Assume that L are the dynamic vertices ("agents") and R are the static vertices ("items"). We refer to the vertices of R as the market.
RANKING algorithm (reminder): it selects a random total ordering of the vertices of R in advance, and when a new vertex j arrives the algorithm matches it to the the unmatched neighbour with the highest rank (highest means the one that occurs earliest according to this ordering).
In this section we prove that RANKING algorithm has a competitive ration of $1 - \frac{1}{e}$. For

our proof, we think of the algorithm in the following way: fix $g : [0,1] \to [0,1]$, $g(x) = e^{x-1}$. Instead of picking a random permutation of the vertices in R, each item j in R picks independently a random number w_j in $[0,1]$ and then computes $g(w_i)$, which is considered as its price. Any agent i in L, upon its arrival, is assigned to the unmatched neighbour j with the lowest $g(w_j)$ (e.g. the cheapest item that is available out of the items she wants). We showed in the lecture that if the following claim holds then RAKING yields a competitive ratio of $1 - \frac{1}{e}$:

Claim 1 *For any ordering of L and for any edge $(i,j) \in E$ the following holds:*

$$\mathop{\mathbb{E}}_{w}\left[u_i + r_j\right] \geq 1 - \frac{1}{e} \tag{1}$$

In this question we prove claim 1.

1. Fix an edge $(i,j) \in E$, a random order of L π, and a prices vector over $R \backslash \{j\}$ $\mathbf{p_{-j}}$. Denote:
   - p the price of the item j' agent i took in $R \backslash \{j\}$ under $\mathbf{p_{-j}}$, set $p = 1$ if there is no such j'.
   - $w := g^{-1}(p)$.
   - for any price p_j of j (i.e. $p_j = g(w_j)$) denote $\mathbf{p} = (\mathbf{p_{-j}}\ , p_j)$.
   - $u_i^{\pi,\mathbf{p}}$ the utility of agent i under π, $\mathbf{p}$.

   Prove:

   (i) $u_i^{\pi,\mathbf{p}} \geq 1 - p$ for any p_j. (i.e., the utility of agent i can't decrease if the market contains more items that she wants).

   (ii) If $p_j < p$ then j is matched under $\pi, \mathbf{p}$. (i.e., adding to the market a cheaper item j than what i took in $R \backslash \{j\}$ assures that j will be taken).

2. Use the $(i), (ii)$ above to end the proof of claim 1 (hint: calculate $\mathbb{E}_w\left[u_i\right]$ and $\mathbb{E}_w\left[r_j\right]$ separately and then use linearity of expectation).